THE ANALYST.

VOL. VIII. SEPTEMBER, 1881. NO. 5.

ON THE ELEMENTARY THEORY OF ERRORS.

BY E. L. DE FOREST.

OUR methods of estimating the error in the result of any combination of independent observations rest upon a well known elementary theorem, demonstrated in some such manner as this. Denote by x_1 and x_2 the true values of two independently observed quantities of the same kind, measured lines for example, and let their sum or difference, taken positively or negatively at pleasure, be

$$X = \pm x_1 \pm x_2. \tag{1}$$

If a single measurement of each gives x_1 and x_2 with the errors $\varDelta_1$ and $\varDelta_2$, which may happen to be either positive or negative, these will produce in X an error $\varDelta$ such that

$$X+\varDelta = \pm(x_1+\varDelta_1) \pm (x_2+\varDelta_2). \tag{2}$$

Subtracting (1) from (2) and squaring the result, we get

$$\varDelta^2 = \varDelta_1^2+\varDelta_2^2 \pm 2\varDelta_1\varDelta_2, \tag{3}$$

where the doubtful sign is + or — according as the signs of x_1 and x_2 in (1) are like or unlike. If n observations of x_1 and x_2 are taken, we have n such equations, and adding them all together and dividing their sum by n, using the brackets [ ] to denote summation, we get

$$\frac{[\varDelta^2]}{n} = \frac{[\varDelta_1^2]}{n} + \frac{[\varDelta_2^2]}{n} \pm 2\frac{[\varDelta_1\varDelta_2]}{n}. \tag{4}$$

Here the first three terms are the squares of the *quadratic mean errors* of X, x_1 and x_2. Denote these q. m. errors by M, μ_1 and μ_2. It may be presumed that the mean of the squares of the errors of a single quantity will not vary much, whatever the number of observations may be, provided it is large, and we may suppose that the number n is very large, so that M, μ_1 and μ_2 have their limiting values. In other words, μ_1 for instance will

represent the square root of the mean of the squares of all the values which the accidental error Δ_1 can possibly have, from the nature of the method of observation employed, each possible value being taken a number of times proportional to the probability of its occurrence.

The two systems of possible errors Δ_1 and Δ_2 are not supposed to be necessarily alike, for x_1 and x_2 may have been measured by two distinct methods. The last term in (4) will disappear if we make the plausible assumption that the errors are distributed symmetrically in either direction, so that positive and negative errors of x_1 of given amount are equally likely to occur; and so also in the case of x_2. Then each positive product $\Delta_1 \Delta_2$ is offset by a negative one equal in amount and equally likely to occur, so that

$$[\Delta_1 \Delta_2] = 0, \tag{5}$$

and (4) is reduced to

$$M^2 = \mu_1^2 + \mu_2^2, \tag{6}$$

a result which has been compared to the geometrical proposition respecting the "square on the hypothenuse", owing to its similar form and its important character as a basis of further investigations.* It holds good approximately even when the number n of observations is not very large, and not the same for x_1 as for x_2. If the observed quantities are three in number, so that

$$X' = \pm x_1 \pm x_2 \pm x_3,$$

the q. m. error of X' may be regarded as that of the sum or difference of the two independently observed quantities X and x_3, and is therefore given by the formula

$$\begin{aligned} M'^2 &= M^2 + \mu_3^2 \\ &= \mu_1^2 + \mu_2^2 + \mu_3^2. \end{aligned} \tag{7}$$

The theorem can thus be extended to four or any number of quantities. Each observed x may also be multiplied by a known coefficient. Any actual error of $a_1 x_1$ is a_1 times the actual arror of x_1, it being understood that the quantity observed is x_1 and not $a_1 x_1$, so that in n observations, the q. m. error of $a_1 x_1$ is $a_1 \mu_1$, that of $a_2 x_2$ is $a_2 \mu_2$, and so on. Hence, if we have any linear function u of independently observed quantities x_1, x_2 &c.,

$$u = a_1 x_1 + a_2 x_2 + a_3 x_3 + \&c., \tag{8}$$

where a_1, a_2 &c., may be essentially either + or —, the q. m. error μ of u will be found from the q. m. errors μ_1, μ_2 &c. of x_1, x_2 &c., by the formula

$$\mu^2 = (a_1 \mu_1)^2 + (a_2 \mu_2)^2 + (a_3 \mu_3)^2 + \&c. \tag{9}$$

*"Dieser Satz, welcher ausserliche Aehnlichkeit mit dem Pythagoraischen Satz der Geometrie hat, ist der wichtigste Satz der ganzen Ausgleichungsrechnung". (Jordan, *Vermessungskunde*, p. 10.)

The exactitude of the foregoing demonstration evidently depends on the correctness of the assumption that the possible true errors of any two of the observed quantities are so distributed as to satisfy the condition (5). All writers on the subject, so far as I know, have regarded the errors as true errors, or deviations from the true value of the quantity, and assumed that in any observation, positive and negative errors of equal amount are equally likely to occur. The mode of demonstration I have given is in substance the one usually followed. (See for instance Chauvenet, *Astronomy*, Vol. II, p. 497; also Helmert, *Ausgleichungsrechnung*, p. 43.) Airy adopts a different method. (*Theory of Errors of Observations*, pp. 28 to 33.) Assuming that the errors of the two observed quantities x_1 and x_2 in (1) follow the exponential law of facility

$$y = ce^{-h^2x^2},$$

while their q. m. errors μ_1 and μ_2 are in general unequal, so as to give different values to h, and consequently to c, in the two cases, it is then proved that the law of facility for the errors of X is of the same exponential form, only their squared q. m. error M^2, is equal to the sum of μ_1^2 and μ_2^2. This mode of proof, like the other, evidently presupposes that + and — errors of equal amount are equally probable, since the exponential curve is symmetrical on either side of its origin. Hence the condition (5) must be satisfied, in order to prove the theorem (6) by either method.

Some recent investigations, however, have enabled us to demonstrate a similar theorem in a manner which is free from this restriction.

Let all the possible true errors of the observed quantities, errors in the lengths of the lines x_1 and x_2 for example, be expressed by multiples of a single unit of measure Δx, which may be taken as small as we please, and let m be a whole number, so large that $m\Delta x$ is a limit which the greatest error, positive or negative, will not exceed. Any observed value, that of x_1 for instance, is here supposed to be obtained either by a single measurement or by taking the arith. mean of several measurements.

Let the probabilities of the occurrence of the various possible errors in the observed values of x_1 and x_2, from $-m\Delta x$ to $+m\Delta x$, be represented by values of p' and p'', ranging from p'_{-m} to p'_m for x_1 and from p''_{-m} to p''_m for x_2. Write these probabilities as coefficients, and the corresponding numbers of units of error as exponents of z, in the polynomials

$$p'_{-m}z^{-m} + \ldots + p'_{-1}z^{-1} + p'_0 + p'_1z + \ldots + p'_mz^m, \tag{10}$$
$$p''_{-m}z^{-m} + \ldots + p''_{-1}z^{-1} + p''_0 + p''_1z + \ldots + p''_mz^m. \tag{11}$$

If the observed values of x_1 and x_2 are to be added together to find X, so that

$$X = x_1 + x_2,$$

the error in this value of X may occur by the algebraic addition of any possible error in the value of x_2 to any possible error in the value of x_1, and the probability that any two particular possible errors of x_1 and x_2 will thus occur in combination is the product of their separate probabilities. The probability that the sum of the two errors that do occur will be a given amount, is the sum of the probabilities of all the combinations which would severally produce that amount. Hence all the possible errors in the sum of x_1 and x_2 will be represented by the exponents, and their probabilities will be the coefficients, in the product of the polynomials (10) and (11), which we denote by

$$q_{-2m}z^{-2m}+\ldots+q_{-1}z^{-1}+q_0+q_1z+\ldots+q_{2m}z^{2m}. \qquad (12)$$

The probability that the error of the sum X will be a given quantity $s\varDelta x$, is the coefficient q_s of z^s in this product.

To determine the quadratic mean error of X from the q. m. errors of x_1 and x_2, we employ two general properties of polynomials which, with their application to a class of questions in probability, were set forth by me in the ANALYST during 1880, and had never been published before so far as I know.

In (10) for example, let the coefficients p' represent the weights of material points ranged along an imponderable straight line or axis of X, at intervals equal to $\varDelta x$, and let this axis be a lever turning about the place of p'_0 as a fulcrum. The distance from the fulcrum to the centre of gravity of the system of weights is the lever arm of the system, and its length is

$$i\varDelta x=(-mp'_{-m}-\ldots-1p'_{-1}+0p'_0+1p'_1+\ldots+mp'_m)\varDelta x. \qquad (13)$$

The sum of the products of the weights on one side of the centre of gravity, into their distances from that centre, is equal to the sum of the products on the other side. Hence the lever arm $i\varDelta x$ is the arithmetical mean of all the possible true errors in the observed value of x_1, each possible error being taken with a weight proportional to the probability of its occurrence. Likewise in (11) we denote by $j\varDelta x$ the lever arm of the system of weights p'' about the place of p''_0, and this arm is the arith. mean of all the possible errors in the observed value of x_2. Of course i and j are essentially $+$ or $-$ according as the centres of gravity lie on the $+$ or $-$ side of the fulcrum. By one of the general properties of polynomials, since (12) is the product of (10) and (11), the lever arm $I\varDelta x$ of its coefficients q about the place of q_0 is the algebraic sum of the arms of the two factors, or

$$I\varDelta x=(i+j)\varDelta x. \qquad (14)$$

This lever arm is the arith. mean of all the possible errors in the value of X. Now suppose that the systems of weights in (10), (11) and (12)

revolve around their respective centres of gravity, and let their radii of gyration be e_1, e_2 and E. By a second general property of polynomials, the square of the radius of gyration for the product is equal to the sum of the squares of the radii for the two factors, or

$$E^2 = e_1^2 + e_2^2, \tag{15}$$

a relation of the same form as that in (6). It also has a similar meaning, when we regard e_1, e_2 and E as quadratic mean errors in this modif'd sense, that they are the q. m. deviations of x_1, x_2 and X, not from their true values, but from the arith. means of all their possible values. These arithmetical and quadratic means are formed as already indicated, counting each possible value, and each deviation, a number of times proportional to the probability that such value and deviation will occur. In (10) for example, the sum of all the coefficients being necessarily unity, e_1^2 is the sum of the products formed by mutiplying each weight p' into the square of its dist. from the centre of rotation, and these distances are the possible errors of x_1, in the modified sense. They are analogous to and yet distinct from what are known as residual errors, or deviations from the arith. mean of a number of observations which is very much less than the whole number of possible errors. For want of a name, the mean we are dealing with might be called the *ultimate* arith. mean, and the deviations, ultimate errors. By a well known mechanical theorem, the moment of inertia, and consequently the radius of gyration, of a system, is a minimum when the axis of rotation passes through the centre of gravity. (See for instance *Weisbach's Mech.*) The ultimate mean, or arith. mean of all the possible values of an observed quantity, is equal to the true value of that quantity, plus the lever arm or arith. mean of all its possible true errors. Hence the ultimate mean is the most probable value of the observed quantity, in the sense that it is the value which renders the quadratic mean of all the possible deviations from it a minimum. (Compare Chauvenet, *Astronomy*, II. p. 476.) In accordance with (14), the most probable error $I\Delta x$ of X is the sum of the most probable errors $i\Delta x$ and $j\Delta x$ of x_1 and x_2, and the most probable value of X is the sum of the most probable values of x_1 and x_2.

Suppose now that X is the difference, instead of the sum, of x_1 and x_2, or

$$X = x_1 - x_2.$$

The possible errors of $-x_2$ are the same as those of $+x_2$, but with contrary signs. Their probabilities remain unchanged, so that the errors are represented by the exponents, and their probabilities are the coefficients in the polynomial

$$p''_m z^{-m} + \ldots + p''_1 z^{-1} + p''_0 + p''_{-1} z + \ldots + p''_{-m} z^m, \tag{16}$$

which is (11) with its coefficients in reversed order. The lever arm of the coefficients about the place of p''_0 is the same as before, but with contrary sign. In the case of (11) it was

$$(-mp''_{-m}-\ldots-1p''_{-1}+0p''_0+1p''_1+\ldots+mp''_m)\Delta x = \quad j\Delta x, \quad (17)$$

while for (16) it is

$$(-mp''_m-\ldots-1p''_1+0p''_0+1p''_{-1}+\ldots+mp''_{-m})\Delta x = -j\Delta x. \quad (18)$$

The lever arm for the product of (10) and (16) is the algebraic sum of the arms of the factors, or

$$I\Delta x = (i-j)\Delta x, \quad (19)$$

so that the arith. mean of all the possible values of X is now the difference instead of the sum, of the arith. means of all the possible values of x_1 and x_2. The radius of gyration of the coefficients in (16), about their centre of gravity, remains the same as in (11), for since the distances of each weight p'' from that centre are unchanged except in sign, their squares are entirely unchanged. Thus the radius for the product of (10) and (16) is expressed by the same relation (15) already found for the product of (10) and (11). In other words, the q. m. error of the difference of x_1 and x_2 is the same as that of their sum. And it is evident that by changing the sign of either x_1 or x_2 as in (1), we shall simply change the sign of the lever arm $i\Delta x$ or $j\Delta x$, without affecting the radius of gyration e_1 or e_2.

The reasoning by which these results are obtained may be extended to the sum or difference of three or more observed quantities, by regarding each new quantity as the subject of a second independent observation to be added to or subtracted from the total of the others, as when we derived (7) from (6). The errors in the total, and their probabilities, will be exponents and coefficients in the continued product of the three or more polynomials which represent the probabilities of error in the three or more quantities. Each quantity x may have a known coefficient a, and the q. m. error of ax is a times that of x. Thus the q. m. error e of the linear function

$$u = a_1x_1 + a_2x_2 + a_3x_3 + \&c.,$$

whose coefficients a may be either + or —, is connected with the q. m. errors e_1, e_2 &c. of x_1, x_2 &c. by the relation

$$e^2 = (a_1e_1)^2+(a_2e_2)^2+(a_3e_3)^2+\&c. \quad (20)$$

The arith. means of all the possible true errors of a_1x_1, a_2x_2, a_3x_3 &c. are represented by the lever arms

$$a_1i\Delta x, \qquad a_2j\Delta x, \qquad a_3k\Delta x, \qquad \&c.,$$

and the arith. mean of all the possible errors of u is the sum of the above, or the lever arm

$$I\Delta x = (a_1i + a_2j + a_3k + \&c.)\Delta x, \quad (21)$$

so that the most probable deviation of u from its true value is the algebraic sum of the most probable deviations of a_1x_1, a_2x_2, &c. from their true values.

The result (20) is more general than (9). Having been obtained without introducing any condition such as (5), it will hold good when the elementary or possible errors of the observed quantities are distributed in any manner, unrestricted by that relation to each other. The q. m. errors μ_1, μ_2, &c., in (9) represent deviations from the true values of x_1, x_2 &c., while the modified or ultimate q. m. errors e_1, e_2 in (20) represent deviations from the arith. means of all the possible values of x_1, x_2 &c., these values being weighted according to the probabilities of their occurrence. The true value of an observed quantity is usually unknown and undiscoverable. We may reasonably assume that the arith. mean of a large number of observed values of x_1 for instance, is an approximation to the arith mean of all the possible values of x_1, when each of the latter values is weighted for the probability of its occurrence. But the mean of all the possible values is not the true value, unless the distribution of the possible true errors is restricted so that their arith. mean, the lever arm $i\Delta x$ in (13), becomes zero. There is some *a priori* reason to believe that they will be so restricted in most cases, for + and — errors of equal amount are equally likely to occur for aught we know to the contrary, at least there is usually no reason to expect a preponderance on one side rather than on the other. The difference between the demonstrations of (9) and (20) is interesting chiefly from the theoretical point of view. I think that the new theorems respecting the lever arm and radius of gyration in polynomials and their products, which have been employed here, enable us to demonstrate the principles of the arith. mean and q. m. error with greater generality, clearness and exactness than heretofore, and help to justify the common use of the quadratic mean in preference to any other mean for the purpose of estimating the amount of error, and the universal practice of taking the arith. mean of a number of observations as the standard value of the observed quantity, from which deviations or errors are to be reckoned. The arith. mean of all the possible values is always the most probable value, in the sense of being the one which renders the quadratic mean of all the possible deviations from it a minimum. It is not necessarily the most probable in the sense of being the most likely to occur in a single observation, although it is approximately the most likely to occur as the arith. mean of a large number of observations. (Lagrange, *Oeuvres*, ed. of 1868, Vol, II. p. 199; also ANALYST, Nov. 1880.)

If for example the coefficient p'_2 in (10) were zero, it would be impossible

for the error $2\Delta x$ to occur in any single observation, though it might be the arith. mean of all the possible errors.

In the probability curve

$$y = \frac{hdx}{\sqrt{\pi}} e^{-h^2x^2}, \qquad h^2 = \frac{1}{2kr^2}, \tag{22}$$

the function y, as I have heretofore shown, is in general the limit of the series of coefficients in the expansion of any polynomial to the k power, when k is a large number, or infinite. The interval Δx between successive coefficients is represented by dx at the limit, and this is a unit of measure for the distance x of any coefficient y from the origin, which is at the cent. of grav. of the series, and for the radius of gyration r of the coeff. in the first power, about their c. g. Thus hdx and h^2x^2 are abstract numbers. The curve does not necessarily represent the law of facility of deviation of a single observation from its most probable value, but it does represent the law of facility of deviation of the arith. mean of a large number k of observations of equal weight, from the most probable value of the mean. For y is the probability of a deviation $x = idx$ in the sum of the observations, i being an integer, and hence it is also the probability of a deviation $x \div k$ in their mean. In order that the distances of the coefficients from their centre of gravity may represent deviations in the mean rather than in the sum of the observations, we have only to bring the coefficients or ordinates y closer together, so that the interval or unit Δx or dx is reduced to $\Delta x \div k$ or $dx \div k$. The most probable value of the mean of any number of observations, that is, the value which renders the q. m. deviation of the mean a minimum, is the arith. mean of all the values the mean can have under the given system of elementary or possible errors, and is equal to the arith. mean of all the possible values of a single observation, each possible value being weighted for the probability of its occurrence. The probabilities of the possible errors of a single observation are usually quite unknown, but it is natural to presume that they will generally follow some such law of distribution as the probabilities of error in the sum or the mean of a large number of observations do. In other words, the sum or the mean of a large number k of equally good observations of one quantity may be regarded as the result of a single complex observation, and its proved exponential law of facility of deviation from its most probable value may be taken to represent the most plausible form of the unknown law of facility for any class of observations. It is in this way, as it seems to me, that a proof that the curve (22) represents the limit of the expansion of a polynomial, affords evidence of the general validity of this exponential law of probability. The fact that the origin and vertex of the curve is located at the centre of gravity of the

coefficients in the expansion, shows that the arithmetical mean of all the possible results of the complex observation is not only the most probable result in the sense of rendering the quadratic mean of all the possible deviations from it a minimum, but also in the sense of being the result most likely to occur. We thus reach as a plausible conclusion, a property of the arith. mean which is virtually taken for granted in the well known Gaussian demonstration of the exponential law of probability, which has been so generally followed in elementary treatises. One objection to that demonstration has been that its assumed axiom, that the arith. mean of a number of observed values is the most probable in the sense of being the one most likely to occur, is not really self evident, but stands in need of proof. (Merriman, *Least Squares*, p. 196.)

The limiting form of the expansion of an entire polynomial, when its coefficients are all positive and their sum is unity, is so important in the theory of probability as to make it desirable to have the simplest possible demonstration of it, omitting those considerations which, in my former proof (ANALYST, Sept. '79), arose from the possibility of negative coefficients. Any such polynomial may be written

$$\lambda_{-m}z^{-m}+\ldots+\lambda_{-1}z^{-1}+\lambda_0+\lambda_1 z+\ldots+\lambda_m z^m, \tag{23}$$

by adding terms with zero coefficients if required. Its expansion to the k power may likewise be written

$$l_{-km}z^{-km}+\ldots+l_{-1}z^{-1}+l_0+l_1 z+\ldots+l_{km}z^{km}. \tag{24}$$

From the relation

$$(\lambda_{-m}z^{-m}+\ldots+\lambda_m z^m)^k = l_{-km}z^{-km}+\ldots+l_{km}z^{km},$$

$$\therefore\quad k\log(\lambda_{-m}z^{-m}+\ldots+\lambda_m z^m) = \log(l_{-km}z^{-km}+\ldots+l_{km}z^{km}),$$

which holds good for all values of z, we get by differentiation with respect to z,

$$k(-m\lambda_{-m}z^{-m-1}-\ldots+m\lambda_m z^{m-1})(l_{-km}z^{-km}+\ldots+l_{km}z^{km})$$
$$=(\lambda_{-m}z^{-m}+\ldots+\lambda_m z^m)(-kml_{-km}z^{-km-1}-\ldots+kml_{km}z^{km-1}.) \tag{25}$$

Forming the coefficient of z^{i-1} in the polynomial product in each member, and equating the two to each other by the principle of indeterminate coefficients, we have

$$k(-m\lambda_{-m}l_{i+m}-\ldots+m\lambda_m l_{i-m}) = (i+m)\lambda_{-m}l_{i+m}+\ldots+(i-m)\lambda_m l_{i-m}.$$

In the second member, let that part which does not have the coefficient i be transferred to the first member; then

$$-m\lambda_{-m}l_{i+m}-\ldots+m\lambda_m l_{i-m}=\frac{i}{k+1}(\lambda_{-m}l_{i+m}+\ldots+\lambda_m l_{i-m}). \tag{26}$$

This expresses the relation between the $2m+1$ coeff's λ of the given polynomial, and any group of $2m+1$ coefficients l in the expansion to the k

power, the rank of the middle l of this group, reckoned from l_0, being i. Let the coefficients λ and l be now represented by two series of ordinates referred to a common origin or place of λ_0 and l_0, so that taking the constant interval Δx between ordinates as a unit of measure, the distance of any coefficient from the origin is equal to the corresponding exponent of z. At the limit, k being very large, every l becomes an ordinate y to the limiting curve, and supposing them to be set very near together, Δx is reduced to dx, and the abscissa corresponding to $y = l_i$ will be

$$x = idx. \tag{27}$$

The whole expansion will occupy the interval $(2km+1)dx$, and this will be extended over the whole infinite axis of X if we make k an infinity of the second order. Since a finite number $2m+1$ of consecutive ordinates y will occupy but an infinitesimal interval along the axis of X, we may regard the curve within this interval as sensibly a straight line coinciding with the tangent at its middle point, and the group of coefficients

$$l_{i-m}, \ldots . \; l_i, \ldots . \; l_{i+m}$$

will be represented by the group of ordinates

$$y - mdy, \ldots y, \ldots y + mdy,$$

so that (26) may be written

$$-m\lambda_{-m}(-y - mdy) - \ldots . + m\lambda_m(-y + mdy)$$
$$= \frac{-i}{k+1}\left\{ \lambda_{-m}(y + mdy) + \ldots . + \lambda_m(y - mdy) \right\}.$$

Collecting separately the coefficients of y and dy, using α and β as auxiliary letters, remembering that $\Sigma\lambda = 1$, and giving i its value from (27), we get

$$\left.\begin{array}{l} \alpha = -m\lambda_{-m} - \ldots + m\lambda_m, \quad \beta = (-m)^2\lambda_{-m} + \ldots + m^2\lambda_m, \\ -\alpha y + \beta dy = \dfrac{-xdx}{(k+1)(dx)^2}(y - \alpha dy). \end{array}\right\} \tag{28}$$

This last is the differential equation of the limiting curve, and α, β and $(k+1)(dx)^2$ are constants, the two first being abstract numbers, while the third is a finite area. If we now regard the coefficients λ as the weights of a series of material points ranged along the imponderable axis of X at intervals equal to the unit of measure Δx, or dx at the limit, then αdx is the lever arm of this system about the place of λ_0, and $\beta(dx)^2$ is the square of its radius of gyration about the same point. If λ_0 is at the centre of gravity of the system, α is zero. If some other λ is the place of the c. g., it may be made the origin or coefficient of z^0 by adding or subtracting a constant integer in all the exponents of z in (23), a change which does not alter the coefficients in the expansion. Any λ may be made the middle of a new group of $(2m+1)$ coefficients, the value of m being changed so as to include

the given coefficients and a certain number of others which are zero. For example, in the polynomial of 5 terms,

$$\tfrac{1}{20}(10z^{-2}+4z^{-1}+3+2z+z^2),$$

if we add 1 to each exponent, and prefix two zero terms, we get the polynomial of 7 terms

$$\tfrac{1}{20}(0z^{-3}+0z^{-2}+10z^{-1}+4+3z+2z^2+z^3),$$

where the form of (23) is preserved, but the origin or place of the coeff. of z^0 has been transferred from the 3 to the 4. The significant coefficients, that is, those which are not terminal zeros, remain the same and in the same order, and hence the two expansions to the k power will have their significant coefficients alike, and the limiting curves will be the same, differing only in position relatively to the origin. In any given case, if the origin is placed at any λ we please, (28) is the diff. equation of the limiting curve, provided the lever arm αdx and radius of gyration $\beta(dx)^2$ are understood to refer to this adopted origin, as the fulcrum and axis of rotation. By the law of continuity, if the origin is placed at any point intermediate between two consecutive coefficients in (23), by adding or subtracting a fractional constant in all the exponents of z, a change which does not alter the coefficients in the expansion, then reckoning α and β with reference to this new origin, (28) still holds good. Here we have the means of causing α to disappear, by transferring the origin or place of λ_0 and z^0 to the centre of gravity of the series of coefficients λ. Then in the expansion the centre of gravity remains at this new place of z^0, so that it becomes the point for which $x = 0$ in the limiting curve, the diff. equation reducing to

$$\frac{dy}{y} = \frac{-xdx}{(k+1)\beta(dx)^2},$$

or by taking a new constant $h^2 = 1 \div 2(k+1)\beta(dx)^2$, (29)

$$\frac{dy}{y} = -2h^2xdx.$$

Hence by integration

$$y = ce^{-h^2x^2}. \tag{30}$$

The sum of the coefficients in the expansion is unity, so that

$$\frac{1}{dx}\int_{-\infty}^{+\infty} ydx = 1,$$

which, as is well known, gives c in terms of hdx,

$$c = \frac{hdx}{\sqrt{\pi}},$$

and the final equation of the limiting curve stands as in (22), for since k is large, or infinite, we may write k for $k+1$ in (29). The squared radius of gyration of the coefficients about their centre of gravity is represented by

$$r^2 = \beta(dx)^2$$

for the given polynomial and becomes

$$kr^2 = k\beta(dx)^2$$

for the expansion to the k power, enabling us to compute the value of h in any given case. Owing to the change of $k+1$ to k, (29) gives

$$kr^2 = 1 \div 2h^2,$$

in precise agreement with the well known result

$$\varepsilon^2 = 1 \div 2h^2,$$

where ε is the q. m. error or radius of gyration as found from the curve by integration. (ANALYST, May '79, p. 69.) The article cited discusses the form of the expansion of the binomial $(p + q)^\infty$, or what is the same thing, the form of the series of coefficients in the expansion of $(p + qz)^\infty$. This becomes a special case under our present method, if we suppose all the coefficients in (23) to be zero except $\lambda_0 = p$ and $\lambda_1 = q$.

Likewise in the equation of the probability surface

$$z = \frac{h_1 h_2 dxdy}{\pi} e^{-(h_1^2x^2+h_2^2y^2)}, \qquad h_1^2 = \frac{1}{2kr_1^2}, \qquad h_2^2 = \frac{1}{2kr_2^2}, \tag{31}$$

the function z represents, as I have shown, the limit of the series of coefficients in the expansion of a polynomial of two variables to a power whose exponent k is large, when the free axes of the system of coefficients are taken as axes of X and Y. (ANALYST, March, '81.)

While the surface does not necessarily represent the law of facility of deviation of a single observed point in a plane, from its most probable position, it does represent the law of facility of deviation of the centre of gravity of a large number k of similarly observed points, from its most probable place. The function z is the probability that the sums of the deviations in the X and Y directions will be $x = idx$ and $y = jdy$ respectively, i and j being integers, and this is the probability that their arith. mean will be $x \div k$ and $y \div k$. If the ordinates z are set closer together, so that dx and dy are reduced to $dx \div k$ and $dy \div k$, they will represent the law of facility of deviation of the centre of gravity of the k observed points, from its most probable place. The most probable position of the centre of gravity is the centre of gravity of all the positions it can possibly occupy under the given system of elementary or possible errors, and the same as the c. g. of all the possible positions of a single observed point; possible positions being always weighted for the prob'ty of their occurrence. Practically, in the absence of any definite knowledge respecting the probabilities of the various possible errors of a single observation of a point, it seems most natural to presume that their distribution will resemble that of the prob's of deviation in the c. g. of a large number of similarly observed points of error. In this way, as it seems to me, the surface (31) may be held to represent the most plausible law of facility of error in the observed position of a point in a plane.